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Quantitative Finance: Problems and Solutions

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Preface

This book is a collection of exercises in quantitative finance for graduate students in financial markets. After the notations have been introduced and the relevant continuous-time models have been discussed, four main topics are addressed. The first section proposes problems based on one-period markets, where the focus is on the determination of no-arbitrage prices for claims that provide given payoff profiles in complete or incomplete markets. Within the same discrete-time framework, the second section aims at fostering the understanding of optimal mean-variance portfolio choices and the related unconstrained or constrained optimization techniques. The third section relies instead on the continuous-time Black-Scholes representation of financial markets in the presence of market risk. The exercises concern the determination of the equilibrium return and the no-arbitrage price of instruments exposed to such a risk via their payoffs. The fourth section deals with the continuous-time Vasicek model of interest rate risk. The exercises focus on the financial features of the no-arbitrage pricing formula of zero-coupon bonds and on the equilibrium term structure of interest rates.

Notations

Notations for one-period financial markets
(Sections 1 and 2)

Timing Investors face two trading dates only, namely $t = 0$ and $t = 1$. At time $t = 0$, investors choose their investment strategy, investing in $N+1$ non-dividend-paying securities, for which we use the index j, with $j = 0, \ldots, N$. At time $t = 1$ they receive the liquidation value of their strategy.

Riskless security The security with $j = 0$ represents a riskless security. $B(0) \equiv 1$ denotes the time-0 price of the riskless security. $B(1) \equiv 1 + r$ denotes its time-1 price. The quantity r is the risk-free return, with $r \geq 0$.

Risky securities For the N securities with $j > 0$, $S_j(0)$ denotes their time-0 price. $\widetilde{S}_j(1)$ is a random variable that denotes their time-1 price, with $j = 1, \ldots, N$.

Uncertainty By time 1, the market uncertainty will resolve in one of K possible states of the world. ω_k indicates the generic k-th state of the world at time 1. The ω_k's are relevant economic/financial scenarios. Ω indicates the set of all states of the world, i.e. $\Omega = \{\omega_1, \ldots, \omega_K\}$. $S_j(1)(\omega_k)$ indicates the time-1 price of the j-th security in scenario ω_k.

Payoff matrix The payoff matrix M has $K + 1$ rows and $N + 1$ columns. Each column j of M represents the time-0 cashflows from buying (row 1) or the time-1 cashflows from having bought (the other rows) 1 unit of the j-th security $(j = 0, 1, ..., N)$:

$$M \equiv \begin{bmatrix} -1 & -S_1(0) & -S_2(0) & \cdots & -S_N(0) \\ 1+r & S_1(1)(\omega_1) & S_2(1)(\omega_1) & \cdots & S_N(1)(\omega_1) \\ 1+r & S_1(1)(\omega_2) & S_2(1)(\omega_2) & \cdots & S_N(1)(\omega_2) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1+r & S_1(1)(\omega_K) & S_2(1)(\omega_K) & \cdots & S_N(1)(\omega_K) \end{bmatrix} \quad \begin{matrix} \text{time } 0 \\ \text{time } 1 \\ \text{time } 1 \\ \vdots \\ \text{time } 1 \end{matrix}$$

Portfolio strategies A portfolio strategy is identified by the column vector ϑ, which is composed of the portfolio positions in the $N+1$ securities. ϑ_0 is the portfolio position in the riskless security. $\vartheta_1, \vartheta_2, ..., \vartheta_N$ are the portfolio positions in the risky securities. The portfolio positions represent the units of each security bought, or sold short, at time 0.

Cashflows of investment strategies At time 0, the quantity $V_\vartheta(0)$ is the initial cost an investor must face to set up the investment strategy ϑ. The corresponding initial cashflow received (if positive) or paid (if negative) by the investor is $f_\vartheta(0) = -V_\vartheta(0)$. The final proceeds are represented with a random variable $\widetilde{V}_\vartheta(1)$. $V_\vartheta(1)(\omega_k)$ is the time-1 cashflow in the state ω_k from liquidating the investment strategy ϑ.

Security returns The (total) returns of the securities are measured at time 1. They are denoted with $R_B(1)$ for the riskless security $(j = 0)$ and with the random variables $\widetilde{R}_{S_j}(1)$ for the risky securities ($j = 1, \ldots, N$):

$$R_B(1) = \frac{B(1) - B(0)}{B(0)} = r$$

$$\widetilde{R}_{S_j}(1) = \frac{\widetilde{S}_j(1) - S_j(0)}{S_j(0)} \quad \text{for} \quad j = 1, ..., N.$$

Portfolio shares The portfolio share (or portfolio weight) ζ_j is the fraction of initial wealth W_0 devoted by the strategy ϑ to the security j:

$$\zeta_0 = \frac{\vartheta_0 B(0)}{W_0}$$

$$\zeta_j = \frac{\vartheta_j S_j(0)}{W_0} \quad \text{for} \quad j = 1, ..., N.$$

Strategy returns The total return of a strategy with portfolio shares ζ (and portfolio positions ϑ) is denoted with the random variable $\widetilde{R}_\zeta(1)$:

$$\widetilde{R}_\zeta(1) = \frac{\widetilde{V}_\vartheta(1) - V_\vartheta(0)}{V_\vartheta(0)}.$$

Notations for the Black-Scholes model with discussion (Section 3)

Timing Investors face a continuum of trading dates. The current date is t and the next date is $t + \mathsf{d}t$ with $\mathsf{d}t$ being the infinitesimal length of an instantaneous period. Time is usually measured in years so that, if T is the future maturity date of a financial contract ($T > t$), $T - t$ represents the number of years to the contract's expiry.

Riskless security The riskless security (the money account) offers the interest payment $L(t)\, r\mathsf{d}t$ over the next instant (of length $\mathsf{d}t$) on the amount $L(t)$ currently invested. Hence, the instantaeous percentage increment in L is

$$\frac{\mathsf{d}L}{L} = r\mathsf{d}t.$$

The per-annum rate of instantaneous return on the security is r, which is assumed to be constant (r is called the riskfree rate). If there is reinvestment of the instantaneous interest payments up to time T, the terminal value of the investment is $L(T) = L(t)\, e^{r(T-t)}$.

Risky underlying stock The risky underlying security is a stock whose current price is S ($S > 0$) and whose dividend paid over the next instant is $Sq\mathsf{d}t$. The per-annum dividend yield is q, which is assumed to be constant. If there is reinvestment of the instantaneous dividends up to time T, the number of stocks owned goes from $m(t)$ at the current date to $m(T) = m(t)\, e^{q(T-t)}$

at the terminal date. It follows that, if we start at time t with $e^{-q(T-t)}$ shares of the stock and we reinvest the dividends, we end up at time T with exactly one share.

Under the objective probability measure P, the underlying stock price dynamics is

$$\frac{\mathsf{d}S}{S} = E_t^P\left[\frac{\mathsf{d}S}{S}\right] + \sigma \mathsf{d}z^P,$$

where σ is the volatility parameter and $\{z^P\}$ is a Wiener process under P (its instantaneous increment $\mathsf{d}z^P$ is the stock-return innovation). The total instantaneous expected return on the stock is

$$E_t^P\left[\frac{\mathsf{d}S}{S}\right] + q\mathsf{d}t = r\mathsf{d}t + \sigma\lambda\rho\mathsf{d}t,$$

where $\sigma\lambda\rho\mathsf{d}t$ is the instantaneous risk premium. The parameter λ ($\lambda > 0$) is the market price of risk, which is the compensation required for holding one unit of systematic risk, and $\rho\mathsf{d}t$ is the current conditional covariance between the stock-return innovation $\mathsf{d}z^P$ and the systematic-risk innovation $\mathsf{d}z_\lambda^P$ ($\mathsf{d}z_\lambda^P < 0$ makes investors feel unexpectedly worse off). If $\rho > 0$, the stock tends to unexpectedly drop in value exactly when investors feel unexpectedly worse off. Hence, the stock does not provide insurance against systematic risk and stock investors demand the positive per-annum risk premium $\sigma\lambda\rho$ as compensation.

In summary, the stock price dynamics is

$$\mathsf{d}S = S\left(r - q + \sigma\lambda\rho\right)\mathsf{d}t + S\sigma\mathsf{d}z^{P},$$

and the level 0 is an absorbing boundary for the stock price process (the process remains at 0 forever if it starts from there).

Derivative contracts Consider a finite-maturity derivative contract that provides a terminal non-negative payoff only (there are no intermediate payouts). The payoff is a contractually specified function of the underlying stock price prevailing at the maturity date T. The current no-arbitrage price of the derivative contract is the function $V\left(S, t\right)$ of the current stock price S and of the current date t. The function V is assumed to admit the partial derivatives

$$\underbrace{\frac{\partial V}{\partial t} = V_t}_{= \Theta \text{ (THETA)}}, \quad \underbrace{\frac{\partial V}{\partial S} = V_S}_{= \Delta \text{ (DELTA)}}, \quad \text{and} \quad \underbrace{\frac{\partial^2 V}{\partial S^2} = V_{SS}}_{= \Gamma \text{ (GAMMA)}},$$

which are called the contract's Greeks. Ito's Lemma states that

$$\mathsf{d}V = \Theta\mathsf{d}t + \Delta\mathsf{d}S + \frac{1}{2}\Gamma S^2\sigma^2\mathsf{d}t.$$

Hence, the total instantaneous return on the derivative contract is

$$\frac{\mathsf{d}V}{V} = \underbrace{E_t^P\left[\frac{\mathsf{d}V}{V}\right]}_{\text{expected return}} + \underbrace{\frac{\Delta S}{V}\sigma \mathsf{d}z^P}_{\text{unexpected return}},$$

$$E_t^P\left[\frac{\mathsf{d}V}{V}\right] = \frac{1}{V}E_t^P\left[\mathsf{d}V\right] = \frac{1}{V}\left(\Theta + \Delta S\left(r - q + \sigma\lambda\rho\right) + \frac{1}{2}\Gamma S^2\sigma^2\right)\mathsf{d}t.$$

If $\rho > 0$ and the current Δ is positive, derivative-contract investors currently demand a positive risk premium,

$$E_t^P\left[\frac{\mathsf{d}V}{V}\right] - r\mathsf{d}t = \frac{\Delta S}{V}\sigma\lambda\rho\mathsf{d}t, \tag{1}$$

as the derivative contract tends to unexpectedly drop in value exactly when investors feel unexpectedly worse off. By contrast, if $\rho > 0$ and the current Δ is negative, investors currently accept a negative risk premium as the contract does offer insurance against systematic risk. Equation (1) can be rewritten as the Black-Scholes second-order partial differential equation (PDE):

$$\Theta + \Delta S\left(r - q\right) + \frac{1}{2}\Gamma S^2\sigma^2 = Vr.$$

The Black-Scholes PDE is associated with two boundary conditions, which come from what can be stated about $V\left(S, t\right)$ if $t \to T$ (payoff condition) and if $S \to 0$ (absorption-related condition).

Futures contract The current no-arbitrage futures price is $F(S,t)$. If we are at the futures contract's expiry/delivery date $(t \to T)$ and we buy the contract, our payoff is $S - F(S, t \to T)$. There are no costs in buying (or selling) the contract so that, by no arbitrage, $F(S, t \to T) = S$. The futures contract's inception date is the initial date 0 $(0 < t < T)$. The marking-to-market process makes sure that the current total value of having bought a futures contract at the initial date is $\int_0^t e^{r(t-u)} dF(S_u, u)$, where the time u is any date running from the initial date to the current date $(0 \leq u \leq t)$. The cost-of-carry pricing formula

$$F(S,t) = Se^{(r-q)(T-t)}$$

meets the no-arbitrage PDE

$$\Theta_F + \Delta_F S(r-q) + \frac{1}{2}\Gamma_F S^2 \sigma^2 = 0$$

and the two conditions

$$F(S, t \to T) = S \qquad \text{and} \qquad F(S \to 0, t) = 0.$$

Power contract The current no-arbitrage price of the power contract is $W(S,t)$. Given $\alpha \geq 0$,the power contract is a derivative security (without intermediate payouts) characterized by the terminal-payoff condition $W(S, t \to T) = S^\alpha$ and (for $\alpha > 0$) by the condition $W(S \to 0, t) = 0$.

The pricing formula

$$W(S,t) \;=\; S^{\alpha}\; e^{\left(-r+\alpha(r-q)+\frac{1}{2}\alpha(\alpha-1)\sigma^2\right)\,(T-t)}$$

meets the Black-Scholes PDE with the two conditions above specified. The pricing formula exhibits the discounting effect $(-r)$, the scaled underlying-drift effect $(\alpha(r-q))$, and the concavity/convexity effect $(\frac{1}{2}\alpha(\alpha-1)\sigma^2)$. The total instantaneous return on the power contract is

$$\frac{\mathsf{d}W}{W} = \underbrace{(r+\alpha\sigma\lambda\rho)\,\mathsf{d}t}_{\text{expected return}} \quad + \quad \underbrace{\alpha\sigma\mathsf{d}z^P}_{\text{unexpected return}},$$

which comes from applying equation (1) to Ito's Lemma for $W(S,t)$ with $\frac{\Delta S}{W} = \alpha$ (if the underlying price grows by 1% the power contract price grows by α% *coeteris paribus*).

Digital option The current no-arbitrage price of the digital option is $D(S,t)$. Given the strike price K, the digital option is a derivative security (without intermediate payouts) characterized by the terminal-payoff condition $D(S,t \to T) = 1_{\{S \geq K\}}$ and by the condition $D(S \to 0,t) = 0$. Given

$$n(v) = \frac{1}{(2\pi)^{\frac{1}{2}}} e^{-\frac{v^2}{2}} \text{ for any real } v, \quad N(x) = \int_{-\infty}^{x} n(v)\,dv \text{ for any real } x,$$